Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
J. Math. Anal. Appl. 327 (2007) 816–828
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

A characterization of the local dual spaces of a Banach space ☆

Manuel González [a,*], Antonio Martínez-Abejón [b]

[a] *Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cantabria, E-39071 Santander, Spain*
[b] *Departamento de Matemáticas, Facultad de Ciencias, Universidad de Oviedo, E-33007 Oviedo, Spain*

Received 20 March 2006
Available online 30 May 2006
Submitted by William F. Ames

Abstract

A *local dual* of a Banach space X is a closed subspace of X^* that satisfies the properties that the principle of local reflexivity assigns to X as a subspace of X^{**}. Here we introduce a technical property which characterizes the local dual spaces of a Banach space and allows us to show new examples of local dual spaces.

Keywords: Banach space; Local reflexivity; Ultrapower

1. Introduction

In [9], the authors define the *local dual spaces* of a Banach space X as those closed subspaces Z of the dual space X^* such that for every pair of finite dimensional subspaces E of X^* and F of X, and every $\varepsilon > 0$, there exists an operator $L: E \to Z$ which satisfies the following conditions:

(a) $(1-\varepsilon)\|e\| \leqslant \|L(e)\| \leqslant (1+\varepsilon)\|e\|$ for all $e \in E$;
(b) $\langle L(e), x\rangle = \langle e, x\rangle$ for all $e \in E$ and all $x \in F$;
(c) $L(e) = e$ for all $e \in E \cap Z$.

☆ Supported in part by DGI (Spain), Grant MTM2005-03831.
* Corresponding author.
E-mail addresses: gonzalem@unican.es (M. González), ama@uniovi.es (A. Martínez-Abejón).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.04.028

The classical principle of local reflexivity [13], as well as the principle of local reflexivity for ultrapowers [12], can be respectively stated as follows

(d) X is a local dual of X^*;
(e) for every ultrafilter $\mathfrak{U}$, $(X^*)_{\mathfrak{U}}$ is a local dual of the ultrapower $X_{\mathfrak{U}}$.

Sometimes, when no representation of X^* is known, it is possible to find local dual spaces of X that admit a concrete realization. That situation is very well exemplified in the statements (d) and (e) above, but we can find in [3,6,9–11] many other examples. Interest in local dual spaces also stems from the merging of operator semigroup theory in the sense of [1] with research on minimal norming subspaces and locally complemented subspaces, done by Godefroy and Kalton [5,14]. Indeed, it was proved in [8] that, given an operator T and an ultrafilter $\mathfrak{U}$, the kernel of the conjugate operator $(T_{\mathfrak{U}})^*$ (henceforth, denoted $T_{\mathfrak{U}}{}^*$) is finitely dual representable in the kernel of the ultrapower operator $(T^*)_{\mathfrak{U}}$ (henceforth, $T^*{}_{\mathfrak{U}}$), which solves some problems about duality of left-regular, surjective semigroups [1]. In that proof, none of the principles of local reflexivity (d) or (e) is directly applicable since $N(T_{\mathfrak{U}}{}^*)$ is neither an ultrapower of $N(T^*)$ nor a second dual space. The connection of the work of Godefroy and Kalton to local dual spaces can be seen in the following result.

Theorem 1.1. [9] *Let Z be a closed subspace of a dual Banach space X^*. The following statements are equivalent*:

(1) *Z is a local dual of X*;
(2) *there exists an isometric extension operator $L : Z^* \to X^{**}$ such that $L(Z^*) \supset X$*;
(3) *there exists a norm one projection P on X^{**} such that $Z^\perp = N(P)$ and $X \subset R(P)$*;
(4) *there exists a norm-one projection Q on X^{***} with $R(Q) = Z^{\perp\perp}$ and $N(Q) \subset X^\perp$.*

This theorem is very useful from a theoretical point of view because it provides us with global characterizations of local dual spaces. Moreover, it has been applied in [10] in order to prove that $L_\infty(\mu, X)^*$ contains a canonical copy of $L_1(\mu, X^*)$ which is a local dual of $L_\infty(\mu, X)$. Its power is also noticeable in order to identify closed subspaces Z of X^* which are not local duals of X despite X^* is finitely representable in Z (for instance, checking if $Z^\perp$ is not complemented in X^{**}). Nevertheless, Theorem 1.1 has not yielded any new proofs of results like (d) or (e) (actually, the proof of (3) $\Rightarrow$ (1) given in [9] needs the classical principle of local reflexivity).

Here, developing some ideas introduced in [16], we introduce the strict polar property as a test to check whether a closed subspace Z of X^* is a local dual of X. As an application, we prove that for every operator $T : X \to Y$ and every ultrafilter $\mathfrak{U}$, the kernel $N(T^*{}_{\mathfrak{U}})$ is a local dual of $Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})}$. Note that $(Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})})^* \equiv N(T_{\mathfrak{U}}{}^*)$. Our result, applied to the zero operator on X, gives the principle of local reflexivity stated in (e).

We also show that, given a local dual Z of X, the dual space X^* is a $\mathcal{L}^1$-space (respectively, a $\mathcal{L}^\infty$-space) if and only if Z is a $\mathcal{L}^1$-space (respectively, a $\mathcal{L}^\infty$-space). In particular this is true for the kernels of $N(T_{\mathfrak{U}}{}^*)$ and $N(T^*{}_{\mathfrak{U}})$. It is not difficult to find examples of operators T with $R(T)$ non-closed for which $N(T_{\mathfrak{U}}{}^*)$ is a $\mathcal{L}^1$-space or a $\mathcal{L}^\infty$-space. Since in this case $N(T^*{}_{\mathfrak{U}})$ is neither an ultrapower nor a dual space (see [7]), we get new examples of $\mathcal{L}^1$-spaces and $\mathcal{L}^\infty$-spaces.

In the paper X and Y are Banach spaces, B_X the closed unit ball of X, S_X the unit sphere of X, and X^* the dual of X; X^n denotes the Cartesian product $X \times \overset{n}{\cdots} \times X$, where n is a positive integer. For a subset $A \subset X$, we denote

$$A^{\perp} := \{ f \in X^*\colon \langle f, x\rangle = 0 \text{ for every } x \in A \}.$$

We denote by $\mathcal{B}(X,Y)$ the space of all (bounded linear) operators from X into Y. Given an operator $T \in \mathcal{B}(X,Y)$, $N(T)$ and $R(T)$ are the range and the kernel of T, and T^* is the conjugate operator of T. Since we deal in this paper with ranges of operators, we will mention explicitly when a subspace is norm closed.

Given a number $\varepsilon > 0$, an operator $T \in \mathcal{B}(X,Y)$ is called *ε-isometry* if it satisfies $1-\varepsilon < \|Tx\| < 1+\varepsilon$ for all $x \in S_X$. A Banach space X is said to be *finitely representable in Y* (X f.r. in Y, for short) if for each $\varepsilon > 0$ and each finite dimensional subspace M of X there is an ε-isometry $T : M \to Y$.

We denote by $\mathbb{N}$ the set of all positive integers, and define $\mathbb{N}^* := \{0\} \cup \mathbb{N}$. An ultrafilter $\mathfrak{U}$ on a set I is *countably incomplete* if there is a countable partition $\{I_n\}_{n=1}^{\infty}$ of I such that $I_n \notin \mathfrak{U}$ for all $n \in \mathbb{N}$. Every infinite set admits a countably incomplete ultrafilter [12]. All ultrafilters occurring in this paper are countably incomplete. Given an ultrafilter $\mathfrak{U}$ on a set I, let $\ell_\infty(I,X)$ be the Banach space of all bounded families $(x_i)_{i\in I}$ in X endowed with the supremum norm, and let $N_{\mathfrak{U}}(X)$ be the closed subspace of all $(x_i) \in \ell_\infty(I,X)$ which converge to 0 following $\mathfrak{U}$. The *ultrapower of X following* $\mathfrak{U}$ is defined as the quotient space

$$X_{\mathfrak{U}} := \ell_\infty(I,X) \,/\, N_{\mathfrak{U}}(X).$$

The element of $X_{\mathfrak{U}}$ including as a representative the family $(x_i) \in \ell_\infty(I,X)$ is denoted by $[x_i]$, and its norm in $X_{\mathfrak{U}}$ is given by $\|[x_i]\| = \lim_{\mathfrak{U}} \|x_i\|$. The ultrapower $X_{\mathfrak{U}}$ contains an isometric copy of X generated by the constant families of $\ell_\infty(I,X)$. We identify this copy with X. Accordingly, every operator $T \in \mathcal{B}(X,Y)$ admits an extension $T_{\mathfrak{U}} \in \mathcal{B}(X_{\mathfrak{U}}, Y_{\mathfrak{U}})$ given by $T_{\mathfrak{U}}([x_i]) = [Tx_i]$. Given a family $(C_i)_{i\in I}$ of subsets of X, we denote

$$(C_i)_{\mathfrak{U}} := \{ [c_i] \in X_{\mathfrak{U}}\colon \exists J \in \mathfrak{U} \text{ such that } \forall i \in J,\ c_i \in C_i \}.$$

The ultrapower $(X^*)_{\mathfrak{U}}$ is a closed subspace of $(X_{\mathfrak{U}})^*$. Actually, $[f_i] \in {X^*}_{\mathfrak{U}}$ is identified with the element $f \in (X_{\mathfrak{U}})^*$ given by $f([x_i]) := \lim_{\mathfrak{U}} f_i(x_i)$. With this identification, the conjugate operator ${T_{\mathfrak{U}}}^*$ is an extension of the ultrapower ${T^*}_{\mathfrak{U}}$ and $N({T^*}_{\mathfrak{U}})$ is a closed subspace of $N({T_{\mathfrak{U}}}^*)$ for any operator $T : X \to Y$. The identity $(X^*)_{\mathfrak{U}} = (X_{\mathfrak{U}})^*$ holds if and only if X is superreflexive [12]; the kernel $N({T^*}_{\mathfrak{U}})$ equals $N({T_{\mathfrak{U}}}^*)$ if and only if T is supertauberian [8]. We refer to [12] for additional information about ultrapowers.

2. The strict polar property

Let X be a Banach space, let E and Z be closed subspaces of X^*, and let F be a closed subspace of X. For an operator $L : F \to Z$ we will consider the following conditions:

(I) $\langle L(e), x\rangle = \langle e, x\rangle$ for all $e \in E$ and all $x \in F$.
(II) $L(e) = e$ for all $e \in E \cap Z$.

We will say that $L : E \to Z$ *satisfies* (I) *or* (II) *with respect to F*.

Definition 2.1. [9] A closed subspace Z of X^* is said to be a *local dual of X* if for every couple of finite dimensional subspaces E of X^* and F of X, and every $\varepsilon > 0$, there is an ε-isometry $L : E \to Z$ that satisfies (I) and (II) with respect to F.

Given an operator $L \in \mathcal{B}(X,Y)$, we denote by L^n the operator in $\mathcal{B}(X^n,Y^n)$ that maps $(x_i)_{i=1}^n$ to $(Lx_i)_{i=1}^n$. Given a Banach space X, every finite scalar matrix $A=(a_{ij})_{i=1}^{k}{}_{j=1}^{l}$ of order $k\times l$ induces an operator $A_X \in \mathcal{B}(X^l,X^k)$ defined by $A_X((x_j)_{j=1}^l) = (\sum_{j=1}^l a_{ij}x_j)_{i=1}^k$. Such an operator is called *matricial*.

Proposition 2.2. *Given a finite scalar matrix A of order $k\times l$ and a Banach space X, the following properties hold*:

(a) *The matricial operator A_{X^*} is the conjugate of the matricial operator $(A^*)_X$, where A^* denotes the conjugate matrix of A.*
(b) *Given a closed subspace Z of X, A_X maps Z^k to Z^l.*
(c) *For every operator $L\in\mathcal{B}(X,Y)$, we have $L^k\circ A_X = A_Y\circ L^l$.*
(d) *Given an ultrafilter $\mathfrak{U}$, we have $(A_X)_{\mathfrak{U}} = A_{(X_{\mathfrak{U}})}$.*

The proofs of properties (a), (b) and (c) are straightforward. For the proof of (d), identify $(X\times X)_{\mathfrak{U}}$ with $X_{\mathfrak{U}}\times X_{\mathfrak{U}}$.

Definition 2.3. We say that a closed subspace Z of X^* has the *strict polar property* if for every $k,l\in\mathbb{N}$, every matricial operator $T:\ell_\infty^l(X^*)\to\ell_\infty^k(X^*)$ and every $z\in\ell_\infty^k(Z)$, the set

$$\ell_\infty^l(Z)\cap T^{-1}(z+B_{\ell_\infty^k(Z)})$$

is $\sigma(\ell_\infty^l(X^*),\ell_1^l(X))$-dense in $T^{-1}(z+B_{\ell_\infty^k(X^*)})$.

The following result characterizes the strict polar property in terms of bounded sets.

Proposition 2.4. *A closed subspace Z of X^* has the strict polar property if and only if for every $k,l\in\mathbb{N}$, every matricial operator $T:\ell_\infty^l(X^*)\to\ell_\infty^k(X^*)$ and every $z\in\ell_\infty^k(Z)$, the set*

$$B_{\ell_\infty^l(Z)}\cap T^{-1}(z+B_{\ell_\infty^k(Z)})$$

is $\sigma(\ell_\infty^l(X^*),\ell_1^l(X))$-dense in $B_{\ell_\infty^l(X^*)}\cap T^{-1}(z+B_{\ell_\infty^k(X^*)})$.

Proof. Assume that Z has the strict polar property. Given an operator T as in the statement, define $\tilde{T}:\ell_\infty^l(X^*)\to\ell_\infty^l(X^*)\oplus_\infty\ell_\infty^k(X^*)$ by $\tilde{T}(z):=(z,Tz)$. By applying Definition 2.3 on $\tilde{T}$ and $\tilde{z}=(0,z)\in\ell_\infty^l(X^*)\oplus_\infty\ell_\infty^k(X^*)$, we get the desired result.

The converse implication is due to the equality $\ell_\infty^l(X^*)=\bigcup_{n=1}^\infty nB_{\ell_\infty^l(X^*)}$. □

Lemma 2.5. *Let X be a Banach space and take any α-net $\{x_i\}_{i\in I}$ in S_X with $0<\alpha<1$. Thus, for every $x\in S_X$, there is a sequence $(x_{i_n})_{n=1}^\infty$ in the net and a scalar sequence $(\lambda_n)_{n=1}^\infty$ such that, for every positive integer n,*

(i) $0\leqslant\lambda_n\leqslant\alpha^{n-1}$, *and*
(ii) $\|x-\sum_{m=1}^n\lambda_m x_{i_m}\|<\alpha^n$.

Proof. The choice of the elements λ_n and x_{i_n} is carried out recursively. First, we take $\lambda_1:=1$ and select x_{i_1} so that $\|x-x_{i_1}\|<\alpha$. Let us assume that $\{\lambda_1,\dots,\lambda_{n-1}\}$ and $\{x_{i_1},\dots,x_{i_{n-1}}\}$ have been already chosen satisfying conditions (i) and (ii). Then we take

$$\lambda_n:=\left\|x-(\lambda_1x_{i_1}+\cdots+\lambda_{n-1}x_{i_{n-1}})\right\|<\alpha^{n-1}.$$

If $\lambda_n = 0$, this recursive procedure ends by setting $\lambda_p = 0$ for all $p \geqslant n$. If $\lambda_n \neq 0$, we select x_{i_n} so that $\|\lambda_n^{-1}(x - \lambda_1 x_{i_1} - \cdots - \lambda_{n-1} x_{i_n}) - x_{i_n}\| < \alpha$, so we get

$$\|x - \lambda_1 x_{i_1} - \cdots - \lambda_{n-1} x_{i_{n-1}} - \lambda_n x_{i_n}\| < \alpha \lambda_n < \alpha^n. \qquad \square$$

Lemma 2.6. *Let E be a closed subspace of a Banach space X, $\{x_i\}_{i\in I}$ an α-net in S_E with $0 < \alpha < 1$. Let $\delta > 0$ and $L : E \to X$ a bounded operator such that $1 - \delta \leqslant \|L(x_i)\| \leqslant 1 + \delta$ for all $i \in I$. Then L is a $(\alpha + \delta)(1 - \alpha)^{-1}$-isometry.*

Observe that, given $\varepsilon > 0$, if α and δ are small enough, then L is an ε-isometry.

Proof. Let $x \in S_E$. By Lemma 2.5, there is a scalar sequence $(\lambda_n)_{n=1}^{\infty}$ and a sequence $(x_{i_n})_{n=1}^{\infty}$ in the net $\{x_i\}_{i\in I}$ such that $x = \sum_{n=1}^{\infty} \lambda_n x_{i_n}$ and $0 \leqslant \lambda_n < \alpha^{n-1}$. Thus

$$\|L(x)\| \leqslant \sum_{n=1}^{\infty} \lambda_n \|L(x_{i_n})\| \leqslant \frac{1+\delta}{1-\alpha} = 1 + \frac{\alpha+\delta}{1-\alpha}. \tag{1}$$

Moreover, we can choose x_j in the net so that $\|x - x_j\| < \alpha$. Thus, by (1),

$$\|L(x)\| \geqslant \|L(x_j)\| - \|L\| \cdot \|x - x_j\| \geqslant 1 - \delta - \frac{1+\delta}{1-\alpha}\alpha = 1 - \frac{\alpha+\delta}{1-\alpha}. \qquad \square$$

Notice that the linear span of $\{x_i\}_{i\in I}$ needs not equal E. Therefore, the condition of boundedness of L in Lemma 2.6 is not superfluous.

Lemma 2.7. *Let X be a Banach space with finite dimension n, and V a subspace of X with dimension $n - k$. Then there exists a biorthogonal system $(x_i, f_i)_{i=1}^n$ of X such that $\|x_i\| = 1$ and $\|f_i\| \leqslant 1 + \sqrt{n}$ for all $i \in \{1, \ldots, n\}$, and $V = \operatorname{span}\{x_j\}_{j=k+1}^n$.*

Proof. By Theorem 4.18 in [4], there exists a projection $P : X \to X$ such that $R(P) = V$ and $\|P\| \leqslant \sqrt{n}$. Let $W := N(P)$. By Auerbach's lemma [15, Proposition 1.c.3], there are two biorthogonal systems $(x_i, h_i)_{i=1}^k$ and $(x_i, h_i)_{i=k+1}^n$ of W and V respectively such that $\|x_i\| = \|h_i\| = 1$ for all $i \in \{1, \ldots, n\}$. Consider Hahn–Banach extensions $g_i \in X^*$ of every h_i and define $f_i := g_i \circ P$ for all $i \in \{k+1, \ldots, n\}$ and $f_i := g_i \circ (I_X - P)$ for all $i \in \{1, \ldots, k\}$. It is immediate that $(x_i, f_i)_{i=1}^n$ is a biorthogonal system of X and that $\|x_i\| = 1$ and $\|f_i\| \leqslant 1 + \sqrt{n}$ for all $i \in \{1, \ldots, n\}$. $\square$

Let us apply these results to characterize the local dual spaces of a Banach space.

Theorem 2.8. *A closed subspace Z of X^* has the strict polar property if and only if it is a local dual of X.*

Proof. Assume that Z is a local dual of X. Let $z = (z_i)_{i=1}^k \in \ell_\infty^k(Z)$, and $T : \ell_\infty^l(X^*) \to \ell_\infty^k(X^*)$ a matricial operator. Given $(f_i)_{i=1}^l \in T^{-1}(z + B_{\ell_\infty^k(X^*)})$, we must show that every w^*-neighborhood $\mathcal{V}$ of $(f_i)_{i=1}^l$ meets $\ell_\infty(Z) \cap T^{-1}(z + B_{\ell_\infty^k(Z)})$. In order to do that, choose $0 < \theta < 1$ and $(g_i)_{i=1}^l \in T^{-1}(z + \theta B_{\ell_\infty^k(X^*)})$ so that $(g_i)_{i=1}^l \in \mathcal{V}$. We take a finite subset $\{x_{ij}\}_{i=1}^{l}{}_{j=1}^{m}$ in X so that

$$\mathcal{V} \supset \{(h_i)_{i=1}^l : |\langle g_i - h_i, x_{ij}\rangle| < 1,\ 1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant m\}.$$

Let $F := \operatorname{span}\{x_{ij} : 1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant m\}$ and $E := \operatorname{span}\{g_i, z_j : 1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant k\}$.

Take $\varepsilon > 0$ so that $\theta(1+\varepsilon) < 1$. Thus, by the hypothesis of local duality, there exists an ε-isometry $L : E \to Z$ such that $L(e) = e$ for all $e \in E \cap Z$ and $\langle Lf, x\rangle = \langle f, x\rangle$ for all $f \in E$ and all $x \in F$. Hence $(Lg_i)_{i=1}^l \in \mathcal{V} \cap \ell_\infty^l(Z)$. Moreover, since $L^k((z_i)_{i=1}^k) = (z_i)_{i=1}^k$, Proposition 2.2 yields

$$T \circ L^l\big((g_i)_{i=1}^l\big) - (z_i)_{i=1}^k = L^k\big(T\big((g_i)_{i=1}^l\big) - (z_i)_{i=1}^k\big),$$

so

$$\big\|T \circ L^l\big((g_i)_{i=1}^l\big) - (z_i)_{i=1}^k\big\| \leqslant \|L\| \big\|T\big((g_i)_{i=1}^l\big) - z\big\| < (1+\varepsilon)\theta < 1.$$

Hence $(Lg_i)_{i=1}^l \in \mathcal{V} \cap T^{-1}(z + B_{\ell_\infty^k(Z)}) \neq \emptyset$, and the proof of the direct implication is done.

For the converse, let us assume that Z has the strict polar property, let $E \subset X^*$ and $F \subset X$ be finite dimensional subspaces, let $\varepsilon > 0$ and find an ε-isometry $L : E \to Z$ satisfying clauses (I) and (II).

Let $n = \dim E$ and $n - k = \dim E \cap Z$. By Lemma 2.7, E has a biorthogonal system $(y_r, h_r)_{r=1}^n$ such that $\|y_r\| = 1$ and $\|h_r\| \leqslant 1 + \sqrt{n}$ for all $r \in \{1, \dots, n\}$, and $E \cap Z = \operatorname{span}\{y_r\}_{r=k+1}^n$. Obviously, an operator $L \in \mathcal{B}(E, Z)$ satisfying condition (II) must be of the form

$$Le := \sum_{r=1}^k h_r(e) v_r + \sum_{r=k+1}^n h_r(e) y_r. \tag{2}$$

Let us find suitable vectors $v_1, \dots, v_k$ so that L is also an ε-isometry satisfying clause (I). In order to proceed, given any pair of real numbers $0 < \alpha < 1$ and $0 < \beta$, we take

a finite α-net $\{e_i\}_{i=1}^M$ in S_E,
a finite system $\{u_i\}_{i=1}^K$ in S_X so that $\|e\| \leqslant (1+\beta)\sup_{1\leqslant i\leqslant K}\langle e, u_i\rangle$ for all $e \in E$,
a basis $\{x_i\}_{i=1}^N$ in F.

Let $\lambda_{ir} := \langle h_i, e_r\rangle$ for all $i \in \{1, \dots, M\}$ and all $r \in \{1, \dots, n\}$, so $|\lambda_{ir}| \leqslant 1 + \sqrt{n}$ and

$$e_i = \sum_{r=1}^n \lambda_{ir} y_r \quad \text{for each } i \in \{1, \dots, M\}.$$

Let us consider the vector

$$y := -\left(\sum_{r=k+1}^n \lambda_{ir} y_r\right)_{i=1}^M \in \ell_\infty^M(Z)$$

and the operators $U : \ell_\infty^k(X^*) \to \ell_\infty^M(X^*)$ and $S : \ell_\infty^k(X^*) \to \mathbb{K}^{kN+kK}$ defined by

$$U\big((f_s)_{s=1}^k\big) := \left(\sum_{s=1}^k \lambda_{is} f_s\right)_{i=1}^M \quad \text{and} \quad S\big((f_s)_{s=1}^k\big) := \big(\langle f_r, x_i\rangle, \langle f_t, u_j\rangle\big)_{r=1}^k{}_{i=1}^N{}_{t=1}^k{}_{j=1}^K.$$

Notice that $\|U\| \leqslant n(1+\sqrt{n})$, that U is conjugate because it is a matricial operator (see Proposition 2.2), and that S is also conjugate since the elements x_i and u_j belong to X.

Consider now the sets

$$D := B_{\ell_\infty^k(X^*)} \cap U^{-1}(y + B_{\ell_\infty^M(X^*)}) \quad \text{and} \quad C := D \cap \ell_\infty^k(Z).$$

By the strict polar property, C is w^*-dense in D. Therefore, since

$$\left\|U\big((y_s)_{s=1}^k\big)-y\right\|=\left\|(e_i)_{i=1}^M\right\|=1,$$

it follows that $(y_s)_{s=1}^k \in D=\overline{C}^{w^*}$. Moreover, since S is a w^*-continuous finite-rank operator, it follows $S(\overline{C}^{w^*})\subset\overline{S(C)}$. So, given any $\gamma>0$, there exist $(c_s)_{s=1}^k\in C$ and $(b_s)_{s=1}^k\in\gamma B_{\ell_\infty^k(Z)}$ such that

$$S\big((y_s)_{s=1}^k\big)=S\big((b_s)_{s=1}^k\big)+S\big((c_s)_{s=1}^k\big).$$

Let us adopt $v_s:=b_s+c_s$ for all $s\in\{1,\dots,k\}$ in the definition of L given in (2). First, realize that the identity $S((y_s)_{s=1}^k)=S((v_s)_{s=1}^k)$ yields

$$\langle y_s,x_j\rangle=\langle v_s,x_j\rangle\quad\text{for all } s\in\{1,\dots,k\}\text{ and all } j\in\{1,\dots,N\},$$

so $\langle L(e),x\rangle=\langle e,x\rangle$ for all $e\in E$ and all $x\in F$, fulfilling clause (I).

Finally, for every element $e_i\in\{e_i\}_{i=1}^M$, the norm $\|L(e_i)\|$ can be bounded above and below as follows. On the one hand,

$$\begin{aligned}\left\|L(e_i)\right\|&=\left\|\sum_{s=1}^k\lambda_{is}(b_s+c_s)+\sum_{s=k+1}^n\lambda_{is}y_s\right\|\\&\leqslant\left\|\left(\sum_{s=1}^k\lambda_{is}c_s+\sum_{s=k+1}^n\lambda_{is}y_s\right)_{i=1}^M+\left(\sum_{s=1}^k\lambda_{is}b_s\right)_{i=1}^M\right\|\\&\leqslant\left\|U\big((c_s)_{s=1}^k\big)-y\right\|+\left\|U\big((b_s)_{s=1}^k\big)\right\|\leqslant 1+n(1+\sqrt{n})\gamma=:\delta(n,\gamma).\end{aligned}$$

On the other hand, since L satisfies clause (I),

$$\left\|L(e_i)\right\|\geqslant\sup_{1\leqslant j\leqslant K}\langle L(e_i),u_j\rangle=\sup_{1\leqslant j\leqslant K}\langle e_i,u_j\rangle\geqslant\frac{1}{1+\beta}=:\delta'(\beta).$$

Therefore, as n and ε are fixed parameters and the values of α, β and γ can be freely chosen, and since $\delta(n,\gamma)\xrightarrow{\gamma\to 0}1$ and $\delta'(\beta)\xrightarrow{\beta\to 0}1$, we may select α, β and γ small enough so that $\delta(n,\gamma)$ and $\delta'(\beta)$ are as close to 1 as we please in order to ensure, by virtue of Lemma 2.6, that L is an ε-isometry. □

3. The kernel $N(T^*_{\mathfrak{U}})$ is a local dual

Given an operator $T\in\mathcal{B}(X,Y)$ and an ultrafilter $\mathfrak{U}$, the kernel $N(T_{\mathfrak{U}}{}^*)$ can be identified with the dual space of the quotient space $Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})}$, but no concrete representation of $N(T_{\mathfrak{U}}{}^*)$ is known. Observe that $N(T^*{}_{\mathfrak{U}})$ is a subspace of $N(T_{\mathfrak{U}}{}^*)$ which is proper precisely when T is not supertauberian [8]. Two main technical difficulties occur in the study of $N(T^*{}_{\mathfrak{U}})$ and $N(T_{\mathfrak{U}}{}^*)$. The first one arises when the subspace $R(T)$ is not closed. In fact, the inclusions $N(T)_{\mathfrak{U}}\subset N(T_{\mathfrak{U}})$, $\overline{R(T_{\mathfrak{U}})}\subset R(T)_{\mathfrak{U}}$ and $R(T)\subset Y\cap R(T_{\mathfrak{U}})\subset\overline{R(T)}$ are always valid, but the reverse inclusions hold if and only if $R(T)$ is closed [7]. Therefore, any attempt of regarding $N(T^*{}_{\mathfrak{U}})$ as an ultrapower space, or as quotient of ultrapowers, does not seem feasible. The second technical difficulty comes up when dealing with intersections of set ultrapowers. In fact, it is necessary to bear in mind that for any pair of set families $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$, $(A_i\cap B_i)_{\mathfrak{U}}$ is always a subset of $(A_i)_{\mathfrak{U}}\cap(B_i)_{\mathfrak{U}}$, but in general, the reverse inclusion does not hold.

Lemma 3.1. *Let $\mathfrak{U}$ be an ultrafilter on a set I and let X be a Banach space. Let $\{D_i^k\colon i \in I,\ k \in \mathbb{N}\}$ be a uniformly bounded family of bounded subsets of X such that, for every $i \in I$, the subset sequence $(D_i^k)_{k=1}^\infty$ is decreasing. For every k, let $D_k := (D_i^k)_\mathfrak{U}$. Thus $\bigcap_{k=1}^\infty D_k = \emptyset$ if and only if $D_k = \emptyset$ for some positive integer k.*

Proof. The 'if' part is trivial. The converse is achieved by means of a diagonalization argument. Indeed, assume that for every $k \in \mathbb{N}$ there exists $\mathbf{w}_k \in D_k$. Thus all subsets D_i^k are non-empty, which allows us to take, for every $k \in \mathbb{N}$, a bounded family $(w_i^k)_{i\in I}$ such that $\mathbf{w}_k = [w_i^k]_i$ and $w_i^k \in D_i^k$ for all $i \in I$. Since $\mathfrak{U}$ is countably incomplete, there exists a decreasing sequence $(J_n)_{n=0}^\infty$ of elements of $\mathfrak{U}$ such that $J_0 := I$ and $\bigcap_{n=0}^\infty J_n = \emptyset$. For every $i \in I$, let us denote by k_i the only positive integer for which $i \in J_{k_i} \setminus J_{k_i+1}$. Thus $\mathbf{w} := [w_i^{k_i}]$ belongs to D_k for all k. In fact, given $k \in \mathbb{N}$, for every $i \in J_k$ we have that $k_i \geqslant k$, so $w_i^{k_i} \in D_i^{k_i} \subset D_i^k$, and therefore $\{i \in I\colon w_i^{k_i} \in D_i^k\} \supset J_k \in \mathfrak{U}$, which proves that $\mathbf{w} \in D_k$ for all k. □

Lemma 3.2. *Let $T \in \mathcal{B}(X,Y)$ be an operator and $\mathfrak{U}$ an ultrafilter on I. Then, for every $\mathbf{y} = [y_i] \in Y_\mathfrak{U}$, the identity $(T^{-1}(y_i + B_Y))_\mathfrak{U} = (T_\mathfrak{U})^{-1}(\mathbf{y} + B_{Y_\mathfrak{U}})$ holds.*

Proof. The inclusion $(T^{-1}(y_i + B_Y))_\mathfrak{U} \subset (T_\mathfrak{U})^{-1}(\mathbf{y} + B_{Y_\mathfrak{U}})$ is trivial. For the converse, let $[x_i] \in (T_\mathfrak{U})^{-1}(\mathbf{y} + B_{Y_\mathfrak{U}})$ and $\lambda := \lim_\mathfrak{U} \|T(x_i) - y_i\| \leqslant 1$. If $\lambda := 0$, obviously $T_\mathfrak{U}([x_i]) = \mathbf{y}$, so $[x_i] \in (T^{-1}(y_i + B_Y))_\mathfrak{U}$. In the case when $\lambda > 0$, we have $J := \{i \in I\colon \|T(x_i) - y_i\| \neq 0\} \in \mathfrak{U}$. Let us define

$$v_i := \lambda \big\|T(x_i) - y_i\big\|^{-1} x_i \quad \text{and} \quad w_i := \lambda \big\|T(x_i) - y_i\big\|^{-1} y_i \quad \text{for all } i \in J, \quad \text{and}$$
$$v_i := 0 \in X \quad \text{and} \quad w_i := 0 \in Y \quad \text{for all } i \in I \setminus J.$$

Thus $[x_i] = [v_i]$ and $[y_i] = [w_i]$. Moreover, $\|T(v_i) - w_i\| \leqslant 1$ for all $i \in J$, so $[x_i] \in (T^{-1}(y_i + B_Y))_\mathfrak{U}$. □

Corollary 3.3. *Given $T \in \mathcal{B}(X,Y)$ and an ultrafilter $\mathfrak{U}$, the following identity holds*:

$$N(T_\mathfrak{U}) = \bigcap_{n=1}^\infty \left(\frac{1}{n} T^{-1}(B_Y)\right)_\mathfrak{U}.$$

Proof. It is enough to apply Lemma 3.2 to the identity $N(T_\mathfrak{U}) = \bigcap_{n=1}^\infty \frac{1}{n}(T_\mathfrak{U})^{-1}(B_{Y_\mathfrak{U}})$. □

Theorem 3.4. *Let $U \in \mathcal{B}(X,Y)$ an operator and $\mathfrak{U}$ an ultrafilter on I. Let $L \in \mathcal{B}(Y,Y)$ be an operator such that $L_\mathfrak{U}$ maps $\overline{R(U_\mathfrak{U})}$ to $\overline{R(U_\mathfrak{U})}$, and denote*

$$\Lambda : \mathbf{x} + \overline{R(U_\mathfrak{U})} \in Y_\mathfrak{U}/\overline{R(U_\mathfrak{U})} \to L_\mathfrak{U}(\mathbf{x}) + \overline{R(U_\mathfrak{U})} \in Y_\mathfrak{U}/\overline{R(U_\mathfrak{U})}$$

the operator induced by $L_\mathfrak{U}$. Then, for every $\mathbf{g} \in N({U^*}_\mathfrak{U})$,

$$\overline{B_{N({U^*}_\mathfrak{U})} \cap \Lambda^{*-1}(\mathbf{g} + B_{N({U^*}_\mathfrak{U})})}^{w^*} = B_{N({U_\mathfrak{U}}^*)} \cap \Lambda^{*-1}(\mathbf{g} + B_{N({U_\mathfrak{U}}^*)}),$$

where w^* represents the $\sigma(N({U_\mathfrak{U}}^*), Y_\mathfrak{U}/\overline{R(U_\mathfrak{U})})$ topology.

Proof. We identify $(Y_\mathfrak{U}/\overline{R(U_\mathfrak{U})})^*$ with $N({U_\mathfrak{U}}^*)$, so $\Lambda^* \in \mathcal{B}(N({U_\mathfrak{U}}^*), N({U_\mathfrak{U}}^*))$. Let us denote

$$A := B_{N({U^*}_\mathfrak{U})} \cap \Lambda^{*-1}(\mathbf{g} + B_{N({U^*}_\mathfrak{U})})$$

and let $\mathbf{f} \in B_{N({U_\mathfrak{U}}^*)} \setminus \overline{A}^{w^*}$. It is enough to show that $\mathbf{f} \notin \Lambda^{*-1}(\mathbf{g} + B_{N({U_\mathfrak{U}}^*)})$.

Let $\mathbf{g} = [g_i]$ and define

$$D_i^k := L^{*-1}\big(g_i + \big(1 + k^{-1}\big)B_{X^*}\big) \quad \text{for all } i \in I \text{ and all } k \in \mathbb{N},$$
$$D_k := \big(D_i^k\big)_{\mathfrak{U}} \cap Y^*{}_{\mathfrak{U}} \quad \text{for all } k \in \mathbb{N}, \quad \text{and}$$
$$D := \bigcap_{k=1}^{\infty} D_k.$$

By Lemma 3.2, it follows $D = (L^*{}_{\mathfrak{U}})^{-1}(\mathbf{g} + B_{X^*{}_{\mathfrak{U}}}) \cap Y^*{}_{\mathfrak{U}}$, so

$$A = D \cap N\big(U^*{}_{\mathfrak{U}}\big). \tag{3}$$

Now, since $\mathbf{f} \in B_{N(U_{\mathfrak{U}}{}^*)} \setminus \overline{A}^{w^*}$, by the Hahn–Banach theorem, there are two real numbers a and b, and $\mathbf{x}_0 \in Y_{\mathfrak{U}}$ such that

$$\langle \mathbf{h}, \mathbf{x}_0 \rangle \leqslant a < b < \langle \mathbf{f}, \mathbf{x}_0 \rangle \quad \text{for all } \mathbf{h} \in A.$$

(In the case when $A = \emptyset$, just take a and b so that $a < b < \langle \mathbf{f}, \mathbf{x}_0 \rangle$.) Let us define

$$V_i := \big\{h \in B_{Y^*}\colon\ b \leqslant \langle h, x_i \rangle\big\} \quad \text{for all } i \in I,$$
$$W := \big\{\mathbf{h} \in B_{N(U^*{}_{\mathfrak{U}})}\colon\ b \leqslant \langle \mathbf{h}, \mathbf{x}_0 \rangle\big\} = (V_i)_{\mathfrak{U}} \cap N\big(U^*{}_{\mathfrak{U}}\big),$$

so $\mathbf{f} \in \overline{W}^{w^*}$ and $W \cap A = \emptyset$.

For every $i \in I$ and every $n \in \mathbb{N}$, we denote $N_i^n := n^{-1}U^{*-1}(B_{X^*})$ and $N_n := (N_i^n)_{\mathfrak{U}}$. Thus, by Corollary 3.3, $N(U^*{}_{\mathfrak{U}}) = \bigcap_{n=1}^{\infty} N_n$, and by application of identity (3),

$$\begin{aligned}
\emptyset = W \cap A &= (V_i)_{\mathfrak{U}} \cap N\big(U^*{}_{\mathfrak{U}}\big) \cap D \\
&= (V_i)_{\mathfrak{U}} \cap \left[\bigcap_{n=1}^{\infty} (N_n \cap D_n)\right] = (V_i)_{\mathfrak{U}} \cap \left[\bigcap_{n=1}^{\infty} \big(\big(N_i^n\big)_{\mathfrak{U}} \cap \big(D_i^n\big)_{\mathfrak{U}}\big)\right] \\
&\supset (V_i)_{\mathfrak{U}} \cap \left[\bigcap_{n=1}^{\infty} \big(N_i^n \cap D_i^n\big)_{\mathfrak{U}}\right] \supset \bigcap_{n=1}^{\infty} \big(V_i \cap N_i^n \cap D_i^n\big)_{\mathfrak{U}}.
\end{aligned}$$

Therefore, by Lemma 3.1, there exists $k \in \mathbb{N}$ such that $(V_i \cap N_i^k \cap D_i^k)_{\mathfrak{U}} = \emptyset$, which yields the existence of $J \in \mathfrak{U}$ such that, for all $i \in J$, $V_i \cap (N_i^k \cap D_i^k) = \emptyset$. Thus,

$$\big\|L^*(v) - g_i\big\| > 1 + \frac{1}{k} \quad \text{for all } i \in J \text{ and all } v \in V_i \cap N_i^k.$$

Hence, for every $j \in J$, the theorem of Hahn–Banach provides us with $y_j \in B_Y$ such that

$$\big\langle L^*(v) - g_j, y_j \big\rangle \geqslant 1 + \frac{1}{k} \quad \text{for all } v \in V_j \cap N_j^k. \tag{4}$$

Take $y_i := 0$ if $i \in I \setminus J$ and let $\mathbf{y}_0 := [y_i]$. Besides, notice that

$$W = (V_i)_{\mathfrak{U}} \cap \left(\bigcap_{n=1}^{\infty} N_n\right) \subset (V_i)_{\mathfrak{U}} \cap N_{k+1} = (V_i)_{\mathfrak{U}} \cap \big(N_i^{k+1}\big)_{\mathfrak{U}}. \tag{5}$$

But $(V_i)_{\mathfrak{U}} \cap (N_i^{k+1})_{\mathfrak{U}} \subset (V_i \cap N_i^k)_{\mathfrak{U}}$; indeed, each $\mathbf{a} \in (V_i)_{\mathfrak{U}} \cap N_{k+1}$ has two representatives, $(b_i)_{i \in I}$ and $(c_i)_{i \in I}$, for which there exists $J_{\mathbf{a}} \in \mathfrak{U}$ such that $b_i \in V_i$ and $c_i \in N_i^{k+1}$ for all $i \in$

$J_{\mathbf{a}}$; hence $\{i \in J_{\mathbf{a}}\colon \|U^*(c_i)\| \leqslant 1/(k+1)\} \in \mathfrak{U}$, and since $\|b_i - c_i\| \underset{\mathfrak{U}}{\to} 0$, it follows that $\{i \in I\colon \|U^*(b_i)\| \leqslant 1/n\} \in \mathfrak{U}$, so $\mathbf{a} = [b_i] \in (V_i \cap N_i^k)_{\mathfrak{U}}$. Thus, formula (5) yields

$$W \subset \left(V_i \cap N_i^k\right)_{\mathfrak{U}}.$$

Therefore, for every $\mathbf{w} \in W$, there exists a subset $J_{\mathbf{w}}$ of J and a bounded family $(w_i)_{i\in I}$ so that $J_{\mathbf{w}} \in \mathfrak{U}$, $\mathbf{w} = [w_i]$ and $w_i \in V_i \cap N_i^k$ for all $i \in J_{\mathbf{w}}$. Therefore, formula (4) yields

$$\left\langle \Lambda^*(\mathbf{w}) - \mathbf{g}, \mathbf{y}_0 \right\rangle = \lim_{\mathfrak{U}} \left\langle L^*(w_i) - g_i, y_i \right\rangle \geqslant 1 + \frac{1}{k}.$$

But $\mathbf{f} \in \overline{W}^{w^*}$, so $\Lambda^*(\mathbf{f}) \in \overline{\Lambda^*(W)}^{w^*}$, hence

$$\left\| \Lambda^*(\mathbf{f}) - \mathbf{g} \right\| \geqslant 1 + 1/k > 1,$$

which means $\mathbf{f} \notin \Lambda^{*-1}(\mathbf{g} + B_{N(U_{\mathfrak{U}}^*)})$, such as we wanted to prove. $\square$

Now we can give our main result.

Theorem 3.5. *Let $T : X \to Y$ be an operator and let $\mathfrak{U}$ be an ultrafilter on I. Then the kernel $N({T^*}_{\mathfrak{U}})$ is a local dual of $Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})}$.*

Proof. By Theorem 2.8, we have to show that $N({T^*}_{\mathfrak{U}})$ has the strict polar property as a subspace of $N({T_{\mathfrak{U}}}^*) = (Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})})^*$. Let M be a matrix of order $k \times l$ and consider the induced matricial operator $M_{N({T_{\mathfrak{U}}}^*)} := \Delta : \ell_\infty^k(N({T_{\mathfrak{U}}}^*)) \to \ell_\infty^l(N({T_{\mathfrak{U}}}^*))$. According to Proposition 2.4, we only need to show the next identity for every $\mathbf{g} \in \ell_\infty^l(N({T^*}_{\mathfrak{U}}))$:

$$\overline{B_{\ell_\infty^k(N({T^*}_{\mathfrak{U}}))} \cap \Delta^{-1}(\mathbf{g} + B_{\ell_\infty^l(N({T^*}_{\mathfrak{U}}))})}^{w^*} = B_{\ell_\infty^k(N({T_{\mathfrak{U}}}^*))} \cap \Delta^{-1}(\mathbf{g} + B_{\ell_\infty^l(N({T_{\mathfrak{U}}}^*))}), \tag{6}$$

where w^* denotes the $\sigma(\ell_\infty^k(N({T_{\mathfrak{U}}}^*)), \ell_1^k(Y_{\mathfrak{U}}/\overline{R(T_{\mathfrak{U}})}))$ topology.

The proof is divided into three cases: $k = l$, $k < l$ and $k > l$.

Case $k = l$. Let $U := T^k \in \mathcal{B}(\ell_1^k(X), \ell_1^k(Y))$. Since the map $\phi : \ell_1^k(Y_{\mathfrak{U}}) \to \ell_1^k(Y)_{\mathfrak{U}}$ that sends $([y_i^j]_i)_{j=1}^k$ to $[(y_i^j)_{j=1}^k]_i$ is a bijective isometry that maps $(R(T_{\mathfrak{U}}))^k$ onto $R(U_{\mathfrak{U}})$, the induced operator

$$\Phi : \ell_1^k(Y_{\mathfrak{U}})/\overline{R(T_{\mathfrak{U}})}^k \to \ell_1^k(Y)_{\mathfrak{U}}/\overline{R(U_{\mathfrak{U}})}$$

is also a bijective isometry, so is $\Phi^* : N({U_{\mathfrak{U}}}^*) \to \ell_\infty^k(N({T_{\mathfrak{U}}}^*))$. Consider the matricial operator $L : \ell_1^k(Y) \to \ell_1^k(Y)$ associated to the matrix M^*, and the operator

$$\Lambda : \ell_1^k(Y)_{\mathfrak{U}}/\overline{R(U_{\mathfrak{U}})} \to \ell_1^k(Y)_{\mathfrak{U}}/\overline{R(U_{\mathfrak{U}})}$$

defined by $\Lambda(\mathbf{x} + \overline{R(U_{\mathfrak{U}})}) := L_{\mathfrak{U}}(\mathbf{x}) + \overline{R(U_{\mathfrak{U}})}$. Thus its conjugate, $\Lambda^* : N({U_{\mathfrak{U}}}^*) \to N({U_{\mathfrak{U}}}^*)$, satisfies $\Lambda^* = \Phi^{*-1} \circ \Delta \circ \Phi^*$, hence to prove formula (6) is equivalent to prove

$$\overline{B_{N({U^*}_{\mathfrak{U}})} \cap \Lambda^{*-1}\left(\left(\Phi^*\right)^{-1}(\mathbf{g}) + B_{N({U^*}_{\mathfrak{U}})}\right)}^{\mathsf{w}^*} = B_{N({U_{\mathfrak{U}}}^*)} \cap \Lambda^{*-1}\left(\left(\Phi^*\right)^{-1}(\mathbf{g}) + B_{N({U_{\mathfrak{U}}}^*)}\right), \tag{7}$$

where w^* is the $\sigma(N({U_{\mathfrak{U}}}^*), \ell_1^k(Y)_{\mathfrak{U}}/\overline{R(U_{\mathfrak{U}})})$-topology. But the operator U satisfies all the conditions of Theorem 3.4, so formula (7) holds, which proves the case $k = l$.

The cases $k < l$ and $k > l$ are proved applying the result in the case $k = l$ to the matricial operators

$$A^* : (\mathbf{f}, \mathbf{g}) \in \ell_\infty^k\left(N\left({T_{\mathfrak{U}}}^*\right)\right) \oplus_\infty \ell_\infty^{l-k}\left(N\left({T_{\mathfrak{U}}}^*\right)\right) \to \Delta^*(\mathbf{f}) \in \ell_\infty^l\left(N\left({T_{\mathfrak{U}}}^*\right)\right)$$

and

$$A^* : \mathbf{f} \in \ell_\infty^k\big(N(T_{\mathfrak{U}}{}^*)\big) \to \big(0, \Delta^*(\mathbf{f})\big) \in \ell_\infty^{k-l}\big(N(T_{\mathfrak{U}}{}^*)\big) \oplus_\infty \ell_\infty^l\big(N(T_{\mathfrak{U}}{}^*)\big),$$

respectively. Therefore, we conclude that the kernel $N(T^*{}_{\mathfrak{U}})$ has the strict polar property as a subspace of $N(T_{\mathfrak{U}}{}^*)$. $\square$

Corollary 3.6. *The ultrapower $X^*{}_{\mathfrak{U}}$ is a local dual of $X_{\mathfrak{U}}$.*

Proof. It is enough to apply Theorem 3.5 to the zero operator acting on X. $\square$

Let $1 \leqslant p \leqslant \infty$ and $1 \leqslant \lambda < \infty$. A Banach space X is said to be an $\mathcal{L}_\lambda^p$-space if for any finite dimensional subspace E of X there is another finite dimensional subspace F of X satisfying $E \subseteq F$ and $d(F, \ell_p^n) \leqslant \lambda$, where $n = \dim F$ and d denotes the Banach–Mazur distance [2]. A Banach space is said to be an $\mathcal{L}^p$-space if it is a $\mathcal{L}_\lambda^p$-space for some $\lambda \geqslant 1$.

Observe that for $1 < p < \infty$, the $\mathcal{L}^p$-spaces are super-reflexive. Therefore the following two results are non-trivial only in the cases $p = 1$ and $p = \infty$. In the proof we will apply the duality properties of the $\mathcal{L}^p$-spaces and the stability of these classes of spaces under taking complemented subspaces or taking ultrapowers. We refer to [2] for details.

Proposition 3.7. *Let $p = 1$ or $p = \infty$, and let Z be a local dual space of X. Then Z is a $\mathcal{L}^p$-space if and only if X^* is a $\mathcal{L}^p$-space.*

Proof. Suppose that X^* is a $\mathcal{L}^1$-space. Then X^{**} is a $\mathcal{L}^\infty$-space. By Theorem 1.1, Z^* is isomorphic to a complemented subspace of X^{**}. Then Z^* is a $\mathcal{L}^\infty$-space, hence Z is a $\mathcal{L}^1$-space.

Conversely, suppose that Z is a $\mathcal{L}^1$-space. It is proved [11] that there is an ultrafilter $\mathfrak{V}$ so that X^* is complemented in $Z_{\mathfrak{V}}$. Since $Z_{\mathfrak{V}}$ is a $\mathcal{L}^1$-space, X^* is also a $\mathcal{L}^1$-space.

The proof of the case $p = \infty$ is identical. $\square$

Corollary 3.8. *Let $p = 1$ or $p = \infty$ and let $T \in \mathcal{B}(X, Y)$. Then $N(T^*{}_{\mathfrak{U}})$ is a $\mathcal{L}^p$-space if and only if $N(T_{\mathfrak{U}}{}^*)$ is a $\mathcal{L}^p$-space.*

Proof. It is a direct application of Theorem 3.5 and Proposition 3.7. $\square$

Remark 3.9. Note that by Corollary 3.8, if $R(T)$ is not closed and $N(T_{\mathfrak{U}}{}^*)$ is a $\mathcal{L}^p$-space ($p = 1$ or ∞), then $N(T^*{}_{\mathfrak{U}})$ is a $\mathcal{L}^p$-space which is neither an ultrapower nor a dual space. So Corollary 3.8 gives a procedure to get new examples of $\mathcal{L}^p$-spaces. We include below two elementary examples of operators satisfying these conditions.

Let us denote by J the unit interval $[0, 1]$. Given an ultrafilter $\mathfrak{U}$ on I, we consider the equivalence relation $\sim$ in J^I defined by $(t_i) \sim (s_i)$ if $\{i\colon t_i = s_i\} \in \mathfrak{U}$. The set-theoretic ultrapower of J following $\mathfrak{U}$ is defined as $J^{\mathfrak{U}} := J^I/\sim$. An element of $J^{\mathfrak{U}}$ with representative $(t_i)_{i\in I}$ is denoted $(t_i)^{\mathfrak{U}}$. Given a collection $\{A_i\}_{i\in I}$ of subsets of J, we define $(A_i)^{\mathfrak{U}} := \{(t_i)^{\mathfrak{U}}\colon t_i \in A_i\}$. The collection of all sets $(A_i)^{\mathfrak{U}}$, for which every A_i is open, constitutes a basis for a topology in $J^{\mathfrak{U}}$. In the following example, we will consider $J^{\mathfrak{U}}$ endowed with that topology. By $C(J)$ and $L_1(J)$, we denote respectively the spaces of continuous functions on J and of integrable functions with respect to the Lebesgue measure μ on J.

Example 3.10. Given an ultrafilter $\mathfrak{U}$ and the operator $T : C(J) \to C(J)$ that maps every $f(t)$ to $tf(t)$, the kernel $N(T_{\mathfrak{U}}{}^*)$ is a $\mathcal{L}^1$-space.

Proof. Note first that $C(J)_{\mathfrak{U}}$, endowed with the operations $[f_i] + [g_i] := [f_i + g_i]$ and $[f_i] \cdot [g_i] := [f_i \cdot g_i]$, is a Banach algebra. Following the representation theorem for ultrapowers of spaces of continuous functions given in [12, Theorem 4.1], there is a compact Hausdorff space K such that $J^{\mathfrak{U}}$ is homeomorphic to a dense subset of K. Let us identify that subset with $J^{\mathfrak{U}}$. Thus, we can set an isometrical algebra isomorphism H between $C(J)_{\mathfrak{U}}$ and $C(K)$ as follows. For every $\mathbf{f} \in C(J)_{\mathfrak{U}}$, take any representative $(f_i)_{i\in J}$ and consider the continuous mapping $f : J^{\mathfrak{U}} \to \mathbb{R}$ that sends each $(t_i)^{\mathfrak{U}}$ to $\lim_{\mathfrak{U}} f_i(t_i)$. Note that f does not depend on the choice of $(f_i)_{i\in I}$. Now, by the density of $J^{\mathfrak{U}}$ in K, there is a continuous extension $h_{\mathbf{f}} : K \to \mathbb{R}$ of f. It is straightforward to check that the operator $H : C(J)_{\mathfrak{U}} \to C(K)$ that maps each $\mathbf{f}$ to $h_{\mathbf{f}}$ is an isometric algebra isomorphism. Thus $R(H)$ is a closed subalgebra of $C(K)$, and since $R(H)$ separates points and for every $\mathbf{t} \in K$ there is $\mathbf{f} \in C(J)_{\mathfrak{U}}$ so that $h_{\mathbf{f}}(\mathbf{t}) \neq 0$, the Stone–Weierstrass theorem shows that H is surjective.

Now, it is elementary to prove that $L := H \circ T_{\mathfrak{U}} \circ H^{-1}$ is the multiplication operator that sends every $\mathbf{f} \in C(K)$ to $\varphi \cdot \mathbf{f}$, where $\varphi : K \to \mathbb{R}$ is the continuous mapping that sends every $(t_i)^{\mathfrak{U}} \in J^{\mathfrak{U}}$ to $\lim_{\mathfrak{U}} t_i$. Let

$$F := \{t \in K:\ \varphi(t) = 0\}.$$

Thus $\overline{R(L)}$ is the M-ideal $\{\mathbf{g} \in C(K):\ \mathbf{g}|_F = 0\}$, and therefore, $\overline{R(L)}^{\perp} = N(L^*)$ is complemented in $C(K)^*$ [17, III.D]. Now, since $C(K)^*$ is a $\mathcal{L}^1$-space, it follows that $N(L^*)$ is also a $\mathcal{L}_1$-space. So we conclude that $N(T_{\mathfrak{U}}{}^*)$ is a $\mathcal{L}^1$-space. □

Example 3.11. Given an ultrafilter $\mathfrak{U}$ and the operator $T : L_1(J) \to L_1(J)$ that sends every $f(t)$ to $tf(t)$, the kernel $N(T_{\mathfrak{U}}{}^*)$ is a $\mathcal{L}^\infty$-space.

Proof. The conjugate operator $T^* : L_\infty(J) \to L_\infty(J)$ maps every $f(t)$ to $tf(t)$. Thus, $N(T^*{}_{\mathfrak{U}})$ consists of all the elements $\mathbf{f}$ that have a representative $(f_i)_{i\in I}$ such that $\operatorname{supp} f_i \subset [0, \varepsilon_i]$ with $\lim_{\mathfrak{U}} \varepsilon_i = 0$. By Corollary 3.8, it is enough to show that $N(T^*{}_{\mathfrak{U}})$ is a $\mathcal{L}^\infty$-space.

Let E be a finite n-dimensional subspace of $N(T^*{}_{\mathfrak{U}})$ and $0 < \varepsilon < 1$. Let $\{[f_i^1], \dots, [f_i^n]\}$ be a basis of E such that there exists a family of positive numbers $(\varepsilon_i)_{i\in I}$ so that $\lim_{\mathfrak{U}} \varepsilon_i = 0$ and for every $k \in \{1, \dots, n\}$, $\operatorname{supp} f_i^k \subset [0, \varepsilon_i]$ for all i. For every i, let us consider the subspaces $F_i := \operatorname{span}\{f_i^k\}_{k=1}^n$. Since all the spaces $L_\infty([0, \varepsilon_i])$ are isometrical copies of $L_\infty(J)$ and $\dim F_i \leqslant n$ for all i, there is a positive number $m = m(n, \varepsilon)$ and a family $\{G_i\}_{i\in I}$ of m-dimensional subspaces of $L_\infty([0, \varepsilon_i])$ which are $(1+\varepsilon)$-isometric to ℓ_∞^m. Obviously, $(G_i)_{\mathfrak{U}}$ is a subspace of $N(T^*{}_{\mathfrak{U}})$ which contains E and is $(1+\varepsilon)$-isometric to ℓ_∞^m. □

References

[1] P. Aiena, M. González, A. Martínez-Abejón, Operator semigroups in Banach space theory, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 4 (2001) 157–205.

[2] J. Bourgain, New classes of $\mathcal{L}^p$-spaces, Lecture Notes in Math., vol. 889, Springer, Berlin/New York, 1981.

[3] S. Díaz, A local approach to functionals on $L^\infty(\mu, X)$, Proc. Amer. Math. Soc. 128 (2000) 101–109.

[4] J. Diestel, H. Jarchow, A. Tonge, Absolutely Summing Operators, Cambridge Stud. Adv. Math., vol. 43, Cambridge Univ. Press, Cambridge, UK, 1995.

[5] G. Godefroy, N. Kalton, Approximating sequences and bidual projections, Quart. J. Math. Oxford (2) 48 (1997) 179–202.

[6] M. González, An introduction to local duality for Banach spaces, in: Proc. First Int. School of Analysis in Andalucía, World Scientific, 2004, pp. 31–51.
[7] M. González, A. Martínez-Abejón, Ultrapowers and semi-Fredholm operators, Boll. Unione Mat. Ital. Sez. B (7) 11 (1997) 415–433.
[8] M. González, A. Martínez-Abejón, Local reflexivity of dual Banach spaces, Pacific J. Math. 56 (1999) 263–278.
[9] M. González, A. Martínez-Abejón, Local dual spaces of a Banach space, Studia Math. 142 (2001) 155–168.
[10] M. González, A. Martínez-Abejón, Local dual spaces of Banach spaces of vector-valued functions, Proc. Amer. Math. Soc. 130 (2002) 3255–3258.
[11] M. González, A. Martínez-Abejón, Ultrapowers and subspaces of the dual of a Banach space, Glasgow Math. J. 45 (2003) 493–501.
[12] S. Heinrich, Ultraproducts in Banach space theory, J. Reine Angew. Math. 313 (1980) 72–104.
[13] W.B. Johnson, H.P. Rosenthal, M. Zippin, On bases, finite dimensional decompositions and weaker structures in Banach spaces, Israel J. Math. 9 (1971) 488–506.
[14] N. Kalton, Locally complemented subspaces and $\mathcal{L}_p$-spaces for $0 < p < 1$, Math. Nachr. 115 (1984) 71–97.
[15] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces I. Sequence Spaces, Springer, New York, 1977.
[16] A. Martínez-Abejón, An elementary proof of the principle of local reflexivity, Proc. Amer. Math. Soc. 127 (1999) 1397–1398.
[17] P. Wojtaszczyk, Banach Spaces for Analysts, Cambridge Stud. Adv. Math., vol. 25, Cambridge Univ. Press, Cambridge, UK, 1991.